SOLUTIONS TO SELECTED PROBLEMS

SAM WEATHERHOG

1. Lebl Chapter 0 and 1

Exercise 0.3.1. Show that $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$. We show this in two different ways. The first method is overkill for this question, but illustrates the process one usually needs to follow to show that two sets are equal.

Method 1. In order to show the equality of the two sets, we will show that $A\backslash(B\cap C) \subseteq (A\backslash B)\cup(A\backslash C)$ and $(A\backslash B)\cup(A\backslash C) \subseteq A\backslash(B\cap C)$.

To show that $A\backslash(B \cap C) \subseteq (A\backslash B) \cup (A\backslash C)$, we must show that every element of $A\backslash(B \cap C)$ is also an element of $(A\backslash B) \cup (A\backslash C)$. To this end, let $x \in A\backslash(B \cap C)$ be an arbitrary element. By the definition of set difference, $x \in A$ and $x \notin B \cap C$. This implies that either $x \notin B$ or $x \notin C$ (or both). If $x \notin B$, then $x \in A\backslash B$. Similarly, if $x \notin C$, then $x \in A\backslash C$. Hence, in either case, $x \in (A\backslash B) \cup (A\backslash C)$. Thus $A\backslash(B\cap C) \subseteq (A\backslash B)\cup(A\backslash C)$.

Conversely, let $x \in (A\backslash B) \cup (A\backslash C)$ be an arbitrary element. By the definition of set union, $x \in A\backslash B$ or $x \in A\backslash C$ (or both). If $x \in A\backslash B$, then $x \in A$ and $x \notin B$. In this case, $x \notin B \cap C$ and so $x \in A\backslash(B \cap C)$. Similarly, if $x \in A\backslash C$, then $x \in A\backslash(B\cap C)$. Thus $(A\backslash B)\cup(A\backslash C) \subseteq A\backslash(B\cap C)$.

Since $A\backslash(B\cap C) \subseteq (A\backslash B)\cup(A\backslash C)$ and $(A\backslash B)\cup(A\backslash C) \subseteq A\backslash(B\cap C)$ the two sets must in fact be equal. That is, $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$. □

The second method relies on some basic algebraic properties of $\cup, \cap$, and set difference.

Method 2. We start with the left-hand side of the equation and show it is equal to the right-hand side.

$$\begin{aligned} A\backslash(B\cap C) &= A\cap(B\cap C)^c && \text{set difference law} \\ &= A\cap(B^c\cup C^c) && \text{De Morgan's law} \\ &= (A\cap B^c)\cup(A\cap C^c) && \text{distributive law} \\ &= (A\backslash B)\cup(A\backslash C) && \text{set difference law} \end{aligned}$$

□

Exercise 0.3.3. Let $f : A \to B$. Let C, D be subsets of B. Show that

(i) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$;
(ii) $f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$.

Proof. Again, we are trying to show an equality of sets so we will show both containments for each part.

(i) We first show $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$. Suppose that $x \in f^{-1}(C \cap D)$ is an arbitrary element. By the definition of pre-image, this implies that $f(x) \in C \cap D$ and thus $f(x) \in C$ and $f(x) \in D$. Again, by the definition of pre-image, this implies that $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. Hence $x \in f^{-1}(C) \cap f^{-1}(D)$ and $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

To show the other containment, suppose that $x \in f^{-1}(C) \cap f^{-1}(D)$. Then $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. By the definition of pre-image, this implies that $f(x) \in C$ and $f(x) \in D$ and hence $f(x) \in C\cap D$. Thus, $x \in f^{-1}(C\cap D)$ (again, by the definition of pre-image) and $f^{-1}(C)\cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$.

Since both $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$ and $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$, we have $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(ii) We first show $f^{-1}(C^c) \subseteq \big(f^{-1}(C)\big)^c$. Suppose $x \in f^{-1}(C^c)$. By the definition of pre-image, this means that $f(x) \in C^c$ and hence $f(x) \notin C$. Thus, $x \notin f^{-1}(C)$ and so $x \in \big(f^{-1}(C)\big)^c$ by the definition of set complement. So we have $f^{-1}(C^c) \subseteq \big(f^{-1}(C)\big)^c$.

To show the other containment, suppose that $x \in \big(f^{-1}(C)\big)^c$. Then $x \notin f^{-1}(C)$ and so $f(x) \notin C$. This implies that $f(x) \in C^c$ and hence $x \in f^{-1}(C^c)$. Thus $\big(f^{-1}(C)\big)^c \subseteq f^{-1}(C^c)$.

Since both $\big(f^{-1}(C)\big)^c \subseteq f^{-1}(C^c)$ and $f^{-1}(C^c) \subseteq \big(f^{-1}(C)\big)^c$, we have $f^{-1}(C^c) = \big(f^{-1}(C)\big)^c$.

□

Exercise 0.3.4. Let $f : A \to B$. Let C, D be subsets of A. Show that

(i) $f(C \cup D) = f(C) \cup f(D)$;

(ii) $f(C \cap D) \subseteq f(C) \cap f(D)$. Find an example for which equality of the sets $f(C \cap D)$ and $f(C) \cap f(D)$ fails. That is, find an f, A, B, C, and D such that $f(C \cap D)$ is a proper subset of $f(C) \cap f(D)$.

Proof. (i) We are trying to show equality of sets so we will show both $f(C \cup D) \subseteq f(C) \cup f(D)$ and $f(C) \cup f(D) \subseteq f(C \cup D)$.

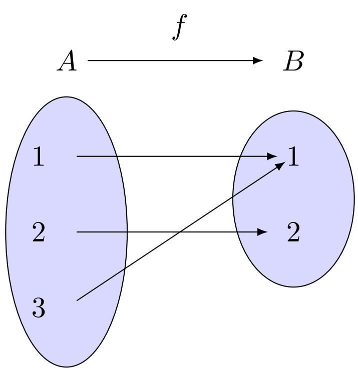
Firstly, let $x \in C \cup D$ be an arbitrary element so that $f(x) \in f(C \cup D)$ is an arbitrary element of $f(C \cup D)$. Since $x \in C \cup D$, $x \in C$ or $x \in D$ (or both). If $x \in C$, then $f(x) \in f(C)$ and hence $f(x) \in f(C) \cup f(D)$. Similarly, if $x \in D$, then $f(x) \in f(D)$ and hence $f(x) \in f(C) \cup f(D)$. In either case, we have $f(C \cup D) \subseteq f(C) \cup f(D)$.

For the second containment, let $y \in f(C) \cup f(D)$ be arbitrary. Then either $y \in f(C)$ or $y \in f(D)$ (or both). If $y \in f(C)$, then there exists $x \in C$ such that $f(x) = y \in f(C)$. In this case, we also have $x \in C \cup D$ so that $y = f(x) \in f(C \cup D)$. Similarly, if $y \in f(D)$, then there exists $z \in D$ such that $f(z) = y \in f(D)$. In this case, we also have $z \in C \cup D$ so that $y = f(z) \in f(C \cup D)$. In either case, $f(C) \cup f(D) \subseteq f(C \cup D)$.

Since both $f(C \cup D) \subseteq f(C) \cup f(D)$ and $f(C) \cup f(D) \subseteq f(C \cup D)$, we have $f(C \cup D) = f(C) \cup f(D)$.

(ii) Since we are only showing one containment, suppose $x \in C \cap D$ is arbitrary so that $f(x) \in f(C \cap D)$ is an arbitrary element. Since $x \in C$ and $x \in D$, we have that $f(x) \in f(C)$ and $f(x) \in f(D)$. Hence $f(x) \in f(C) \cap f(D)$ and thus $f(C \cap D) \subseteq f(C) \cap f(D)$.

It may seem plausible at this point to try and show the containment $f(C) \cap f(D) \subseteq f(C \cap D)$ using the same method of proof as we did for part (i) above. If we try and do this, however, we will run into a problem (because it is not always true that $f(C) \cap f(D) \subseteq f(C \cap D)$). To illustrate this, let $y \in f(C) \cap f(D)$ be arbitrary. Then $y \in f(C)$ and $y \in f(D)$ and hence there exists $x \in C$ and $z \in D$ such that $f(x) = y$ and $f(z) = y$. The problem we have now is that x and z need not be the same element (i.e. there is no reason that $x \in C \cap D$ or $z \in C \cap D$). Although this proof has failed, it should make it clear how to construct a counterexample: we simply need $x \in C \backslash D$ and $z \in D \backslash C$ to map to the same element. There are many ways to do this. One such example is shown below. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1, 2\}$ and $D = \{2, 3\}$. Define $f : A \to B$ by



Now $C \cap D = \{2\}$ and $f(C \cap D) = \{2\}$ but $f(C) = \{1,2\}$ and $f(D) = \{1,2\}$ so $f(C) \cap f(D) = \{1,2\} \neq \{2\}$.

□

Exercise 0.3.8. For each $n \in \mathbb{N}$, define $A_n := \{(n+1)k : k \in \mathbb{N}\}$.

(i) Find $A_1 \cap A_2$;
(ii) Find $\cup_{n=1}^{\infty} A_n$;
(iii) Find $\cap_{n=1}^{\infty} A_n$.

Solution. This question illustrates well the difference between what you "know" and what you can prove.

(i) Firstly we note that $A_1 = \{2, 4, 6, 8 \ldots\}$ is all the multiples of 2 and $A_2 = \{3, 6, 9, 12, \ldots\}$ is all the multiples of 3. At this point, it should be "obvious" that the intersection will be all the multiples of 6. That is, we claim $A_1 \cap A_2 = A_5$. At the moment this is *just* a claim (this is an example of something we "know" but don't have a proof for yet). To prove these two sets are equal, we show containment in both directions.

We first show that $A_1 \cap A_2 \subseteq A_5$. Suppose $x \in A_1 \cap A_2$. Then x is a common multiple of 2 and 3 and hence x must be a multiple of $\text{lcm}(2,3) = 6$. Hence $x = 6k$ for some $k \in \mathbb{N}$ and thus $x \in A_5$.

Now suppose that $x \in A_5$. Then $x = 6k$ for some $k \in \mathbb{N}$. Hence $x = 2(3k) = 3(2k)$ and since both $2k \in bN$ and $3k \in bN$ we have $x \in A_1$ and $x \in A_2$. Thus $x \in A_1 \cap A_2$ and so $A_5 \subseteq A_1 \cap A_2$.

Combining the containments, we have $A_1 \cap A_2 = A_5$.

(ii) Again, it should be 'easy' to come up with a candidate for the union $\cup_{n=1}^{\infty} A_n$. One would expect that every number would be in the union except 1 (since 1 doesn't seem to appear in any set). That is, we claim $\cup_{n=1}^{\infty} A_n = \mathbb{N} \backslash \{1\}$. In order to prove this claim, we need to show that $\cup_{n=1}^{\infty} A_n \subseteq \mathbb{N} \backslash \{1\}$ and $\mathbb{N} \backslash \{1\} \subseteq \cup_{n=1}^{\infty} A_n$.

Since the A_n are defined as subsets of $\mathbb{N}$, to show $\cup_{n=1}^{\infty} A_n \subseteq \mathbb{N} \backslash \{1\}$ we just need to show that 1 doesn't appear in any of the A_n. For a contradiction, suppose that there is some $n \in \mathbb{N}$ with $1 \in A_n$. Then we can write $1 = (n+1)k$ for some $k \in \mathbb{N}$. However, since $n, k \in \mathbb{N}$, the only possibility is $n + 1 = 1$ and $k = 1$. This implies that $n = 0 \notin \mathbb{N}$ (according to the definition of $\mathbb{N}$ in Lebl). This is a contradiction so no such n exists.

For the inclusion $\mathbb{N} \backslash \{1\} \subseteq \cup_{n=1}^{\infty} A_n$, we need to show that every natural number except 1 appears in some A_n. For this, note that for $n \geq 2$, we have $n \in A_{n-1}$ (since $n = (n - 1 + 1) \cdot 1$).

Since $\cup_{n=1}^{\infty} A_n \subseteq \mathbb{N} \backslash \{1\}$ and $\mathbb{N} \backslash \{1\} \subseteq \cup_{n=1}^{\infty} A_n$, we have $\cup_{n=1}^{\infty} A_n = \mathbb{N} \backslash \{1\}$.

(iii) This part is essentially asking "what number is a multiple of every natural number?" Since there is no such number, we expect that $\cap_{n=1}^{\infty} A_n = \emptyset$. In order to show this, we need to show that for each $k \in \mathbb{N}$, there is some A_n for which $k \notin A_n$. Note that for each $n \in \mathbb{N}$ we have $n \notin A_n$ so we are done.

□

Exercise 0.3.13. Prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1},$$

for all $n \in \mathbb{N}$.

Proof. Since we are trying to prove something for all natural numbers, we should try a proof by induction.

For $n = 1$, the left-hand side is $\frac{1}{1 \cdot 2} = \frac{1}{2}$ and the right-hand side is $\frac{1}{1+1} = \frac{1}{2}$ so the result holds in this case.

Suppose that there exists some $k \in \mathbb{N}$ such that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We hope that this will force the result to hold when $n = k+1$. For $n = k+1$, we have the left-hand side given by:

$$\begin{aligned} LHS &= \underbrace{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}}_{\text{first k terms of the sum}} + \underbrace{\frac{1}{(k+1)(k+2)}}_{\text{k+1 term of the sum}} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, && \text{(by inductive hypothesis)} \\ &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \\ &= \frac{k+1}{k+2} \\ &= RHS \end{aligned}$$

We have shown that whenever the result holds for a natural number $n = k$, it will also hold for $n = k+1$. Since the result holds for $n = 1$, by mathematical induction, the result is true for all $n \in \mathbb{N}$. $\square$

2. Lebl Chapter 1 - Ordered Fields, Sup, Inf

Exercise 1.1.1. Let F be an ordered field and $x, y \in F$. Show that if $x < 0$ and $y < z$, then $xy > xz$.

Proof. This is hopefully a familiar property of inequalities. It says that if you take an inequality and multiply through by a negative number you need to flip the direction of the inequality (e.g. $2 < 5$ but multiplying through by -3 we get $-6 > -15$). Since this is not a surprising fact, the main point of this exercise lies in using the basic axioms to prove the result.

Since $y < z$, property (i) of ordered fields tells us that $0 < z - y$ (add $-y$ to both sides). By part (i) of Proposition 1.1.8, since $x < 0$, we have $0 < -x$. Now property (ii) of ordered fields implies that

$$\begin{aligned} & 0 < -x(z-y) \\ \Longrightarrow\ & 0 < -xz + xy \\ \Longrightarrow\ & xz < xy, && \text{by property (i) of ordered fields.} \end{aligned}$$

□

Exercise 1.1.2. Let S be an ordered set. Let $A \subset S$ be a non-empty, finite subset. Show that A is bounded and that $\inf A$ and $\sup A$ both exist and are elements of A.

Proof. Following the hint we use a proof by induction. We induct on the size of the set A. Let $n = |A|$.

For $n = 1$, the set A is bounded (the single element in A is both a lower and upper bound for the set) and $\inf A = \sup A$ are both the only element in A.

Suppose that for $n = k$, the set A is bounded and $\inf A$ and $\sup A$ exist and are in A. Then for $n = k + 1$, consider the subset $A' \subset A$ with one element removed, say $a \in A$. Since A' has k elements, $\inf A'$ exists and is in A'. Let $a' = infA$. If $a' < a$, then $\inf A = \inf A' = a' \in A$. Otherwise, $a < a'$ and so $\inf A = a \in A$. Similarly, $\sup A'$ exists and is in A'. Let $a'' = \sup A'$. If $a'' > a$, then $\sup A = \sup A' = a'' \in A$. Otherwise $a'' < a$ and so $\sup A = a \in A$. □

Exercise 1.1.3. Let $x, y \in \mathbb{F}$ where $\mathbb{F}$ is an ordered field. If $0 < x < y$, show that $x^2 < y^2$.

Proof. Before we start the proof we will make some remarks which hopefully give the reader an idea of how one comes up with these proofs. Firstly, this result should be familiar. The main point of the exercise is using the axioms of an ordered field to prove the result. Note that none of the "multiplication axioms" seem to work at first (since we are multiplying x by x but y by y rather than multiplying through by some element). Rearranging the inequality we have $0 < y^2 - x^2 = (y - x)(y + x)$. Hence, if we can show that $0 < (y - x)(y + x)$ we would be done. Looking through the axioms, it seems property (ii) of ordered fields would work as long as we can show that $y - x > 0$ and $y + x > 0$. This is where our proof starts.

Since $x < y$, by property (i) of ordered fields, $0 < y - x$. Since $0 < y$, by property (i) we have $x < x + y$. Thus $0 < x < x + y$ (this follows from the transitivity of $<$). Hence, by property (ii) of ordered fields

$$\begin{aligned} & 0 < (y-x)(y+x) \\ \Longrightarrow\ & 0 < y^2 - x^2 \\ \Longrightarrow\ & x^2 < y^2, && \text{by property (i).} \end{aligned}$$

□

Exercise 1.1.5. Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. If $b \in A$ show that $b = \sup A$.

Proof. In order to show that $b = \sup A$ we need to show that for any other upper bound, say u, of A, we have $b \le u$. But by the definition of upper bound, we must have $a \le u$ for all $a \in A$. Since $b \in A$ this implies that $b \le u$ and so b is the least upper bound. That is $b = \sup A$. □

Exercise 1.1.6. Let S be an ordered set. Let $A \subset S$ be a non-empty subset that is bounded above. Suppose $\sup A$ exists and $\sup A \notin A$. Show that A contains a countably infinite subset. In particular, A is infinite.

Proof. This is a fairly obvious statement. We first consider a more concrete example. Let S be the set of real numbers and A be the subset (interval) $[0, 1)$. The supremum of A is 1 but $1 \notin A$. Since there is no maximum value of A, we can continually pick numbers closer and closer to 1 (this will give us a countably infinite subset). We now formalize this idea.

Since A is non-empty, pick some $x_0 \in A$. Since $x_0 \in A$, $x_0 \neq \sup A$. Hence there exists $x_1 \in A$ such that $x_0 < x_1$ (otherwise x_0 is an upper bound for A and since $x_0 \in A$, Exercise 1.1.5 implies that $\sup A = x_0$ which is a contradiction). We can now repeat this argument with x_1 in place of x_0. This generates a countably infinite subset $x_0 < x_1 < x_2 < \ldots$. □

Exercise 1.2.3. Let $A \subset \mathbb{R}$ be non-empty. Show that

(i) if $x \in \mathbb{R}$ and A is bounded below, then $\inf(x + A) = x + \inf(A)$;
(ii) if $x > 0$ and A is bounded above, then $\sup(xA) = x \sup(A)$;
(iii) if $x > 0$ and A is bounded below, then $\inf(xA) = x \inf(A)$;
(iv) if $x > 0$ and A is bounded below, then $\sup(xA) = x \inf(A)$;
(v) if $x > 0$ and A is bounded above, then $\inf(xA) = x \sup(A)$;

Proof.

(i) We will show both the inequalities $x+\inf A \leq \inf(x+A)$ and $\inf(x+A) \leq x+\inf A$. Firstly, suppose l is a lower bound for A, i.e. $l \leq a$ for all $a \in A$. Then by property (i) of ordered fields, $x + l \leq x + a$ for all $a \in A$ and hence $x + l$ is a lower bound for $x + A$. In particular, if we take $l = \inf A$, then we have $x + \inf A \leq \inf(x + A)$ (since $\inf(x + A)$ is the greatest lower bound of $x + A$).

Now suppose that l is a lower bound for $x + A$, i.e. $l \leq x + a$ for all $a \in A$. By property (i) of ordered fields, this implies that $l - x \leq a$ for all $a \in A$ and hence $l - x$ is a lower bound for A. In particular, if we take $l = \inf(x+A)$ then we have $\inf(x+A)-x \leq \inf(A)$ (since $\inf(A)$ is the greatest lower bound of A). Applying property (i) of ordered fields, this implies that $\inf(x + A) \leq x + \inf(A)$.

Since we have shown both $x + \inf A \leq \inf(x + A)$ and $\inf(x + A) \leq x + \inf A$, we must have $\inf(x + A) = x + \inf(A)$.

(ii) Suppose that l is an upper bound for A, i.e. $a \leq l$ for all $a \in A$. Then by Proposition 1.1.8 (ii), $xa \leq xl$ for all $a \in A$ and thus xl is an upper bound for xA. In particular, if we take $l = \sup A$ then we have $\sup(xA) \leq x \sup A$.

On the other hand, suppose l is an upper bound for xA, i.e. $xa \leq l$ for all $a \in A$. Since $x > 0$, by Proposition 1.1.8 (v), $1/x > 0$ and so $a \leq l/x$ for all $a \in A$. That is, l/x is an upper bound for A. If we take $l = \sup(xA)$ then we have $\sup A \leq \sup(xA)/x$ which implies that $x \sup A \leq \sup(xA)$.

Since we have shown both $x \sup A \leq \sup(xA)$ and $\sup(xA) \leq x \sup A$ we must have $\sup(xA) = x \sup(A)$.

□

Exercise 1.2.4. Let $x, y \in \mathbb{R}$ and suppose $x^2 + y^2 = 0$. Show that $x = y = 0$.

Proof. By Proposition 1.1.8 (iv), we have $x^2 \geq 0$ and $y^2 \geq 0$. Using property (i) of ordered fields, we can add y^2 to both sides of $0 \leq x^2$ to obtain

$$y^2 \leq x^2 + y^2 = 0.$$

Now $0 \leq y^2 \leq 0$ and hence $y^2 = 0$ and thus $y = 0$ (again by Proposition 1.1.8 (iv)). Similarly, we can add x^2 to the inequality $0 \leq y^2$ to get $0 \leq x^2 \leq 0$ and hence $x = 0$. □

Exercise 1.3.2. Show that

(i) $\max\{x, y\} = \frac{x+y+|x-y|}{2}$;
(ii) $\min\{x, y\} = \frac{x+y-|x-y|}{2}$.

Proof.

(i) There are two cases: either $x \geq y$ or $x < y$. In the first case, the left-hand side is $\max\{x, y\} = x$ and since $|x - y| = x - y$, the right-hand side is

$$\frac{x + y + |x - y|}{2} = \frac{x + y + x - y}{2} = \frac{2x}{2} = x.$$

In the second case, the left-hand side is $\max\{x, y\} = y$ and since $|x - y| = y - x$, the right-hand side is

$$\frac{x + y + |x - y|}{2} = \frac{x + y + y - x}{2} = \frac{2y}{2} = y.$$

(ii) Again, there are two cases: either $x \geq y$ or $x < y$. In the first case, the left-hand side is $\min\{x, y\} = y$ and since $|x - y| = x - y$, the right-hand side is

$$\frac{x + y - |x - y|}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y.$$

In the second case, the left-hand side is $\min\{x, y\} = x$ and since $|x - y| = y - x$, the right-hand side is

$$\frac{x + y - |x - y|}{2} = \frac{x + y - y + x}{2} = \frac{2x}{2} = x.$$

□

Exercise 1.3.3. Find a number M such that $|x^3 - x^2 + 8x| \leq M$ for all $-2 \leq x \leq 10$.

Solution. This is an exercise in applying the triangle inequality. We have

$$|x^3 - x^2 + 8x| \leq |x^3| + |x^2| + |8x| = |x|^3 + |x|^2 + 8|x|.$$

Since $-2 \leq x \leq 10$ we have $|x| \leq 10$ and so

$$|x^3 - x^2 + 8x| \leq |x|^3 + |x|^2 + 8|x| \leq 10^3 + 10^2 + 8 \cdot 10 = 1180.$$

Hence we can take $M = 1180$. Note that this is not the best possible M (the best possible M is 980). □

Exercise 1.3.5. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions (D non-empty).

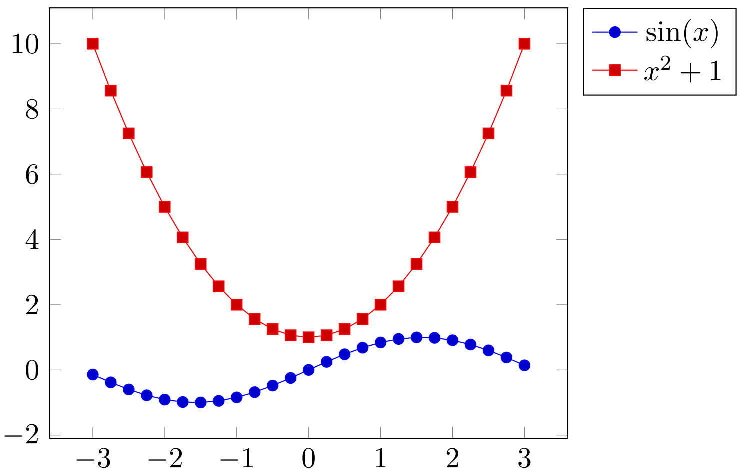
(i) Suppose $f(x) \leq g(y)$ for all $x \in D$ and $y \in D$. Show that

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

(ii) Find a specific D, f, and g such that $f(x) \leq g(x)$ for all $x \in D$, but

$$\sup_{x \in D} f(x) > \inf_{x \in D} g(x).$$

Proof. We first make some remarks that hopefully help with intuition. If $D = \mathbb{R}$ and f and g are familiar functions that you can graph, then the condition in (i) says that the graph of f lies completely below the graph of g. An example of two functions satisfying this would be $f(x) = \sin(x)$, $g(x) = x^2 + 1$:

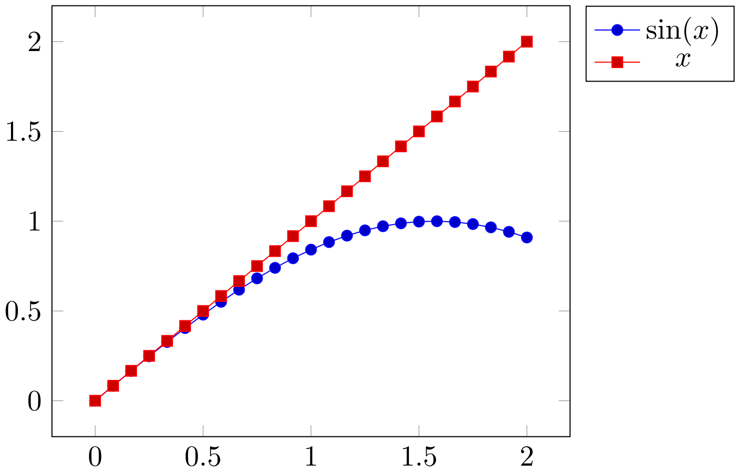


Looking at this graph, the conclusion should be obvious.

The difference between part (i) and part (ii) is that part (ii) only requires f to be "pointwise" less than or equal to g. That is, at each point x, $f(x) \leq g(x)$ but it may be possible to find an x and y such that $f(x) > g(y)$ (which is how we would get the conclusion of part (ii)).

(i) For any given $y \in D$, we have $f(x) \leq g(y)$ for all $x \in D$. Hence $g(y)$ is an upper bound for the set $\{f(x) : x \in D\}$. Therefore $\sup_{x\in D} f(x) \leq g(y)$. Since this holds for all $y \in D$, this implies that $\sup_{x\in D} f(x)$ is a lower bound for the set $\{g(x) : x \in D\}$. Hence $\sup_{x\in D} f(x) \leq \inf_{x\in D} g(x)$.

(ii) There are many examples that will work. One is $D = [0, 2]$, $f(x) = \sin(x)$, $g(x) = x$:



We have $\sup_{x\in D} f(x) = 1$ and $\inf_{x\in D} g(x) = 0$.

$\square$

Exercise 1.3.7. Let D be a non-empty set. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

(i) Show that

$$\sup_{x\in D} \big(f(x) + g(x)\big) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x) \quad \text{and} \quad \inf_{x\in D} \big(f(x) + g(x)\big) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x).$$

(ii) Find an example where we obtain strict inequalities.

3. Lebl Chapter 2

Exercise 2.1.3. Is the sequence $\left\{\frac{(-1)^n}{2n}\right\}$ convergent? If so, what is the limit?

Rough Working. We claim that the sequence converges to 0. We need to show that for every $\epsilon > 0$, we can find $M \in \mathbb{N}$ such that $|x_n - 0| = |x_n| < \epsilon$ for all $n > M$. So we need

$$\left|\frac{(-1)^n}{2n}\right| < \epsilon \implies \frac{1}{2n} < \epsilon \implies \frac{1}{2\epsilon} < n.$$

So we should pick $M = \frac{1}{2\epsilon}$. Since we want $M \in \mathbb{N}$, we can take

$$M = \left\lceil \frac{1}{2\epsilon} \right\rceil.$$

□

We can now write a "clean" proof.

Proof. Given any $\epsilon > 0$, let $M = \left\lceil \frac{1}{2\epsilon} \right\rceil$. Then for any $n > M$, we have

$$\left|\frac{(-1)^n}{2n}\right| = \frac{1}{2n} < \frac{1}{2M} \leq \frac{1}{2\frac{1}{2\epsilon}} = \epsilon.$$

Hence the sequence converges to 0. □

Exercise 2.1.7. Let $\{x_n\}$ be a sequence.

(i) Show that $\lim x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim |x_n| = 0$.
(ii) Find an example such that $|x_n|$ converges but x_n diverges.

Proof. We first note that part (i) is a very strong statement that only works because the limit is 0. In general, we only have one direction: if $\{x_n\}$ converges, then $\{|x_n|\}$ also converges. Part (ii) shows that the converse of this is false in general.

(i) Since this is an if and only if statement, there are two directions to show. Firstly, we show that if $\lim x_n = 0$ then $\lim |x_n| = 0$. Suppose that $\lim x_n = 0$. By the definition of a limit, this means that for every $\epsilon > 0$, we can find $M \in \mathbb{Z}$ such that $|x_n - 0| = |x_n| < \epsilon$ for all $n \geq M$. But this means that for every $\epsilon > 0$, we can find $M \in \mathbb{Z}$ (just use whatever M works above) such that $||x_n| - 0| = |x_n| < \epsilon$ for all $n \geq M$. Hence $\lim |x_n| = 0$.

    Similarly, to show that $\lim |x_n| = 0$ implies that $\lim x_n = 0$, suppose that $\lim |x_n| = 0$. By the definition of a limit, this means that for every $\epsilon > 0$, we can find $M \in \mathbb{Z}$ such that $||x_n| - 0| = |x_n| < \epsilon$ for all $n \geq M$. Again, this immediately gives us what we need since now, for every $\epsilon > 0$, we can find $M \in \mathbb{Z}$ (again, choose the same M that worked above) such that $|x_n - 0| = |x_n| < \epsilon$ for all $n \geq M$.

(ii) There are many sequences that will satisfy this condition. An easy one is the sequence $\{(-1)^n\}$. For this sequence, we have $\{|x_n|\}$ equal to the constant sequence of 1's (i.e. $x_n = 1$ for all n) which clearly converges to 1. The original sequence, however, does not converge.

□

Exercise 2.1.13. Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that

$$\lim_{n\to\infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

Proof. Since the sequence is monotone increasing, $x_n \geq x_k$ for all $n \geq k$. By Theorem 2.1.10, $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ hence $x_k = \sup\{x_n : n \in \mathbb{N}\}$. This implies that $x_n \leq x_k$ for all $n \in \mathbb{N}$ (in particular, for all $n \geq k$). Combining the inequalities we have $x_n = x_k$ for all $n \geq k$. □

Exercise 2.2.3. Prove that if $\{x_n\}$ is a convergent sequence and $m \in \mathbb{N}$ then

$$\lim_{n\to\infty} x_n^m = \big(\lim_{n\to\infty} x_n\big)^m$$

Proof. For $m = 1$ the result is trivial. Suppose that

$$\lim_{n\to\infty} x_n^k = \big(\lim_{n\to\infty} x_n\big)^k,$$

for some $k \in \mathbb{N}$. Then for $m = k+1$, we have

$$\begin{aligned}\lim_{n\to\infty} x_n^{k+1} &= \lim_{n\to\infty} x_n^k x_n \\ &= \big(\lim_{n\to\infty} x_n^k\big)\big(\lim_{n\to\infty} x_n\big) \quad \text{(by Proposition 2.2.5 (iii))} \\ &= \big(\lim_{n\to\infty} x_n\big)^k\big(\lim_{n\to\infty} x_n\big) \quad \text{(inductive hypothesis)} \\ &= \big(\lim_{n\to\infty} x_n\big)^{k+1}.\end{aligned}$$

Hence the result is true by mathematical induction. □

Exercise 2.2.4. Suppose $x_1 := \frac{1}{2}$ and $x_{n+1} := x_n^2$. Show that $\{x_n\}$ converges and find $\lim x_n$.

Proof. The first few terms of the sequence are

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \ldots$$

It appears that the sequence is monotone decreasing and bounded. We need to prove this. We claim that $0 < x_{n+1} < x_n < 1$ for all $n \geq 1$. The proof of this claim is by induction.

For $n = 1$, we have $x_{n+1} = x_2 = \frac{1}{4}$ and $x_1 = \frac{1}{2}$ so the result holds in this case. Suppose that for $n = k$, $0 < x_{k+1} < x_k < 1$. Then by the properties of ordered fields (see Exercise 1.1.3), we have

$$\begin{aligned} &0 < (x_{k+1})^2 < (x_k)^2 < 1^2 \\ \implies &0 < x_{k+2} < x_{k+1} < 1.\end{aligned}$$

Hence the result holds for $n = k+1$. Thus, by induction, the claim is true.

Now by Theorem 2.1.10, the sequence converges. Since $x_{n+1} = x_n^2$, we have

$$\lim x_n = \big(\lim x_n\big)^2.$$

This implies

$$\begin{aligned} 0 &= \big(\lim x_n\big)^2 - \lim x_n \\ &= \lim x_n\big(\lim x_n - 1\big).\end{aligned}$$

Hence $\lim x_n = 0$ or $\lim x_n = 1$. The limit can't be 1 since $x_n \leq \frac{1}{2}$ for all $n \in \mathbb{N}$. Thus $\lim x_n = 0$. □

Exercise 2.3.4. Prove that a bounded sequence $\{x_n\}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}$ converges to x.

Proof. Firstly, suppose that $\{x_n\}$ is a bounded sequence that converges to x. By Theorem 2.3.5 we have $\limsup x_n = \liminf x_n = x$. Proposition 2.3.6 then implies that for any convergent subsequence $\{x_{n_k}\}$,

$$x = \liminf x_n \leq \liminf x_{n_k} \leq \limsup x_{n_k} \leq \limsup x_n = x.$$

Hence $\liminf x_{n_k} = \limsup x_{n_k} = x$ and so $\{x_{n_k}\}$ converges to x. Since $\{x_{n_k}\}$ was arbitrary, this shows that any convergent subsequence converges to x.

For the other direction, suppose that every convergent subsequence $\{x_{n_k}\}$ converges to x. By Theorem 2.3.4, there exist convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that

$$x = \lim x_{n_k} = \limsup x_n, \quad x = \lim x_{m_k} = \liminf x_n.$$

Hence $\limsup x_n = \liminf x_n = x$ and applying Theorem 2.3.5 we have $\{x_n\}$ is convergent and converges to x.

□

Exercise 2.3.5.

(i) Define $x_n := \frac{(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.
(ii) Define $x_n := \frac{(n-1)(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.

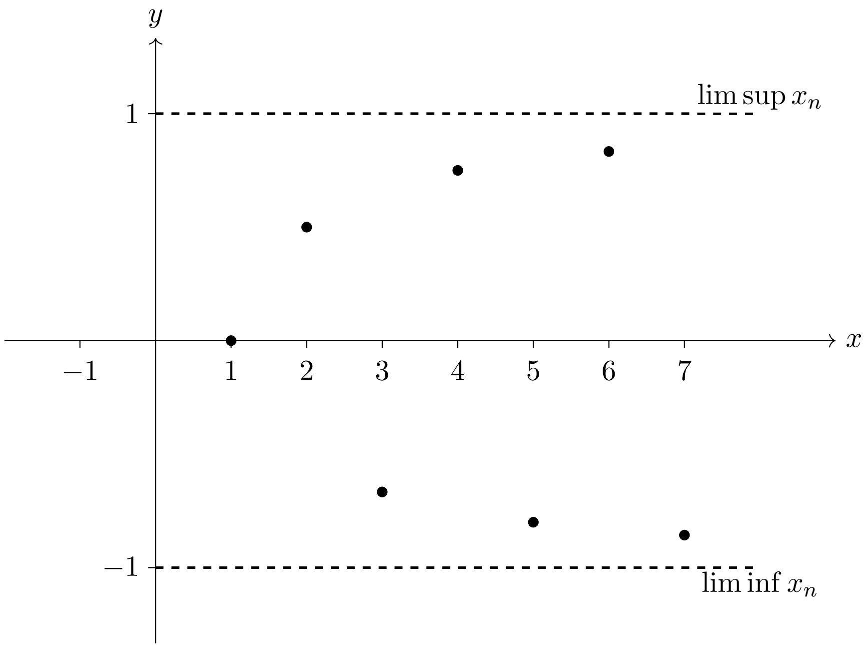
Solution.

(i) The first thing to note is that this sequence converges to 0 (which makes this question a bit cheap). To see this, note that $|x_n| = \frac{1}{n}$ converges to 0 and by Exercise 2.1.7, this implies that x_n converges to 0. Thus, by Theorem 2.3.5 of Lebl, we have $\limsup x_n = \liminf x_n = \lim x_n = 0$.

(ii) The first few terms of this sequence are

$$\left\{0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots\right\}.$$

We will show that $\inf x_n = -1$ and $\sup x_n = 1$. First, we show what is happening diagrammatically:



We will now make some *non-rigourous* observations based on the diagram above. From the diagram, it appears that the positive terms are approaching 1 (which will turn out to be the $\limsup x_n$) and the negative terms are approaching -1 (which will turn out to be the $\liminf x_n$). In fact, we can see that $\sup x_n = 1$ and $\inf x_n = -1$ (i.e. 1 is the supremum of the entire sequence and -1 is the infimum of the entire sequence). To calculate $\limsup x_n$ and $\liminf x_n$ we consider the sequences $a_n := \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$. That is, we successively "chop off" terms from our sequence and look at the sup and inf of what is left. Looking at the diagram above, we can see that removing the first few terms (i.e. removing the first few dots) will not affect the sup or inf; that is, we expect that $a_n = \{1\}$ (the constant sequence of 1's) and $b_n = \{-1\}$ (the constant sequence of -1's).

We now make the above observations rigourous. For the supremum, we need only consider terms of the form $\frac{n-1}{n} = 1 - \frac{1}{n}$ (since the other terms are negative). For $n \in \mathbb{N}$ we have $n \geq 1 \implies \frac{1}{n} \leq 1 \implies 0 \leq 1 - \frac{1}{n}$. Also, since $0 < \frac{1}{n}$, we have $0 > -\frac{1}{n} \implies 1 > 1 - \frac{1}{n}$. Hence $0 \leq 1 - \frac{1}{n} < 1$ and 1 is an upper bound for the sequence. In order to show that 1 is the supremum of the sequence, we need to show that this is the least upper bound. Suppose that there exists $u \in \mathbb{R}$ such that u is an upper bound for the sequence and $0 < u < 1$. Then $0 < 1 - u \implies 0 < \frac{1}{1-u}$. By the Archimedean property, there exists $n \in \mathbb{Z}$ such that $0 < \frac{1}{1-u} < n$. This implies that

$$\frac{1}{1-u} < n \implies 1 - u > \frac{1}{n} \implies 1 - \frac{1}{n} > u.$$

But $1 - \frac{1}{n}$ is an element of the sequence, contradicting the choice of u as an upper bound. Hence no such u exists and thus $\sup\{x_k : k \geq n\} = 1$ for all $n \in \mathbb{N}$.

Similarly, for the infimum, we need only consider the negative terms $\frac{1}{n} - 1$. A calculation similar to the one above shows that $\inf\{x_k : k \geq n\} = -1$ for all $n \in \mathbb{N}$.

The above calculations show that $\limsup x_n = \sup\{x_k : k \geq n\} = 1$ and $\liminf x_n = \inf\{x_k : k \geq n\} = -1$ (hence this sequence does not converge).

□

Exercise 2.3.6. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences such that $x_n \leq y_n$ for all n. Show that

$$\limsup x_n \leq \limsup y_n$$

and

$$\liminf x_n \leq \liminf y_n.$$

Proof. Let $a_n = \sup\{x_k : k \geq n\}$ and $b_n = \sup\{y_k : k \geq n\}$. We claim that $a_n \leq b_n$ for each $n \in \mathbb{N}$. To show this, fix $n \geq 1$. For each $k \geq n$ we have

$$x_k \leq y_k \leq \sup\{y_k : k \geq n\} = b_n.$$

Hence b_n is an upper bound for $\{x_k : k \geq n\}$. By definition of supremum, we have $a_n = \sup\{x_k : k \geq n\} \leq b_n$. Since $a_n \leq b_n$ for eqch $n \in \mathbb{N}$, $\lim a_n \leq \lim b_n$. That is, $\limsup x_n \leq \limsup y_n$. □

Exercise 2.3.7. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.

(i) Show that $\{x_n + y_n\}$ is bounded.

(ii) Show that $\liminf x_n + \liminf y_n \leq \liminf(x_n + y_n)$.

(iii) Find an example where the above inequality is strict.

Proof.

(i) Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exists $X \in \mathbb{R}$ and $Y \in \mathbb{R}$ such that $|x_n| < X$ and $|y_n| < Y$ for all $n \in \mathbb{N}$. But then $|x_n + y_n| \leq |x_n| + |y_n| < X + Y$ for all $n \in \mathbb{N}$. Hence $\{x_n + y_n\}$ is also bounded.

(ii) Since $\{x_n + y_n\}$ is a bounded sequence, by Theorem 2.3.4, there exists a subsequence $\{x_{n_i} + y_{n_i}\}$ that converges to $\liminf(x_n + y_n)$. Now the sequence $\{x_{n_i}\}$ is bounded and hence has a convergent subsequence $\{x_{n_{m_i}}\}$ (by Bolzano-Weierstrass – Theorem 2.3.8). Similarly, $\{y_{n_{m_i}}\}$ has a convergent subsequence, say $\{y_{n_{m_{l_i}}}\}$. Now

$$\begin{aligned}\liminf(x_n + y_n) &= \lim\left(x_{n_{m_{l_i}}} + y_{n_{m_{l_i}}}\right) \\ &= \lim\left(x_{n_{m_{l_i}}}\right) + \lim\left(y_{n_{m_{l_i}}}\right) \\ &\geq \liminf(x_n) + \liminf(y_n),\end{aligned}$$

where the last inequality follows from Proposition 2.3.6.

(iii) Let $\{x_n\} = \{(-1)^n\}$ and $\{y_n\} = \{(-1)^{n+1}\}$. Then $\liminf x_n = \liminf y_n = -1$ but $\{x_n + y_n\}$ is the constant sequence $\{0\}$. Hence $\liminf x_n + \liminf y_n = -2$ and $\liminf(x_n + y_n) = 0$.

□

Exercise 2.4.1. Prove that $\left\{\frac{n^2-1}{n^2}\right\}$ is Cauchy using the definition of Cauchy sequences.

Rough Working. We need to show that for every $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$ and $m \geq M$ we have

$$|x_n - x_m| < \epsilon.$$

Firstly, note that

$$x_n - x_m = \frac{n^2-1}{n^2} - \frac{m^2-1}{m^2} = \frac{n^2m^2 - m^2 - m^2n^2 + n^2}{n^2m^2} = \frac{n^2 - m^2}{n^2m^2} = \frac{1}{m^2} - \frac{1}{n^2}.$$

Using the triangle inequality we have

$$|x_n - x_m| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| \leq \left|\frac{1}{m^2}\right| + \left|\frac{1}{n^2}\right|.$$

If we choose $n, m > M$ then $\frac{1}{n}, \frac{1}{m} < \frac{1}{M}$ and we have

$$\left|\frac{1}{m^2}\right| + \left|\frac{1}{n^2}\right| < \frac{1}{M^2} + \frac{1}{M^2} = \frac{2}{M^2}.$$

We want to choose our M so that $\frac{2}{M^2} < \epsilon$. Rearranging for M, we see that a good choice would be $M = \sqrt{\frac{2}{\epsilon}}$. Since Lebl's definition requires M to be a natural number, we can take $M = \left\lceil \sqrt{\frac{2}{\epsilon}} \right\rceil \geq \sqrt{\frac{2}{\epsilon}}$. We can now write out the proof pretending we didn't do any of this work. □

Proof. Given any $\epsilon > 0$, let $M = \left\lceil \sqrt{\frac{2}{\epsilon}} \right\rceil$. Then for any $n, m \geq M$, we have

$$\begin{aligned}
|x_n - x_m| &= \left|\frac{1}{m^2} - \frac{1}{n^2}\right| \\
&\leq \left|\frac{1}{m^2}\right| + \left|\frac{1}{n^2}\right| \\
&< \frac{1}{M^2} + \frac{1}{M^2} = \frac{2}{M^2} \\
&= \frac{2}{\left\lceil \sqrt{\frac{2}{\epsilon}} \right\rceil^2} \\
&\leq \frac{2}{\sqrt{\frac{2}{\epsilon}}^2} = \frac{2}{\frac{2}{\epsilon}} \\
&= \epsilon.
\end{aligned}$$

Thus we have shown that $|x_n - x_m| < \epsilon$ and hence the sequence satisfies the definition of Cauchy. □

4. Lebl Chapter 3 - Continuous Functions

Exercise 3.1.1. Find the following limits or prove that the limit does not exist.

(i) $\lim_{x\to c} \sqrt{x}$, for $c \geq 0$;
(ii) $\lim_{x\to c} x^2 + x + 1$, for any $c \in \mathbb{R}$;
(iii) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$;
(iv) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$;
(v) $\lim_{x\to 0} \sin(x)\cos\left(\frac{1}{x}\right)$.

Proof.

(i) We claim that the limit is $\sqrt{c}$.

Proof using the definition of a limit: We need to show that for any $\epsilon > 0$ we can find a $\delta > 0$ such that whenever $|x - c| < \delta$, $|f(x) - \sqrt{c}| < \epsilon$. We need to relate $|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}|$ to $|x - c|$. Note that $|\sqrt{x} - \sqrt{c}||\sqrt{x} + \sqrt{c}| = |x - c|$. Since $x, c \geq 0$, using the triangle inequality we have

$$|\sqrt{x} - \sqrt{c}| \leq |\sqrt{x}| + |\sqrt{c}| = |\sqrt{x} + \sqrt{c}|.$$

Now let $\delta = \epsilon^2$. The above inequality implies that when $|x - c| < \delta$,

$$|\sqrt{x} - \sqrt{c}|^2 \leq |\sqrt{x} - \sqrt{c}||\sqrt{x} + \sqrt{c}| = |x - c| < \delta = \epsilon^2.$$

Hence, when $|x - c| < \delta$, we have $|\sqrt{x} - \sqrt{c}| < \epsilon$. Note that the δ we have chosen depends only on ϵ (and not on c). Thus, this shows that the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is uniformly continuous (though continuity and uniform continuity have not been covered in the text at this point).

Alternative Proof: We give an alternative proof using sequential limits. By Lemma 3.1.7, if we can show that for every sequence $\{x_n\}$ that converges to c, the sequence $\{\sqrt{x_n}\}$ converges to $\sqrt{c}$, then we are done. Let $\{x_n\}$ be a sequence converging to c. By Proposition 2.2.6, we have

$$\lim \sqrt{x_n} = \sqrt{\lim x_n} = \sqrt{c}$$

as required.

(ii)

(iii) We have $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ for all $x \in \mathbb{R} - \{0\}$. Hence $-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$ for all $x \in \mathbb{R} - \{0\}$. Now since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, Corollary 3.1.11 (squeeze theorem) implies that $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$.

(iv) Note that $\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) = \frac{1}{2}\sin\left(\frac{2}{x}\right)$. We will now use Lemma 3.1.7 to show that the limit does not exist. That is, we will find a sequence $\{x_n\}$ that converges to 0 but for which $\left\{\frac{1}{2}\sin\left(\frac{2}{x}\right)\right\}$ does not converge. Consider the sequence $\left\{\frac{2}{\pi n + \pi/2}\right\}$. This sequence converges to 0 but the sequence:

$$\left\{\frac{1}{2}\sin\left(\frac{2}{\frac{2}{\pi n+\pi/2}}\right)\right\} = \left\{\frac{1}{2}\sin\left(\pi n + \pi/2\right)\right\} = \left\{\frac{1}{2}\cos(\pi n)\right\} = \left\{\frac{1}{2}(-1)^n\right\},$$

does not converge. Hence Lemma 3.1.7 implies that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)$ does not exist.

(v)

□

Exercise 3.1.5. Let $A \subset S$. Show that if c is a cluster point of A, then c is a cluster point of S.

Proof. Since c is a cluster point of A, by Proposition 3.1.2, there exists a convergent sequence $\{x_n\}$ with $x_n \neq c$, $x_n \in A$ and $\lim x_n = c$. Since $A \subset S$, we have $x_n \in S$ and so Proposition 3.1.2 also implies that c is a cluster point of S. □

Exercise 3.2.1. Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous using the definition.

The proof of this already appears in the text as Example 3.1.5.

Exercise 3.2.10. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Proof. We know that $f(x) = g(x)$ for rational x so we only need to show that this is also true for irrational x. We first show that given any irrational number x, we can find a sequence of *rational* numbers $\{x_n\}$ such that $\lim x_n = x$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (this is the reason everything works here!), given any real numbers $a < b$ we can find a rational number r such that $a < r < b$ (see Theorem 1.2.4). Now let $x \in \mathbb{R}$ be irrational and consider $a_n = x - \frac{1}{n}$ and $b_n = x + \frac{1}{n}$. By Theorem 1.2.4, for each $n \in \mathbb{N}$, we can find a rational number x_n such that $a_n < x_n < b_n$. Since $\{a_n\}$ and $\{b_n\}$ converge to x, the squeeze theorem (Lemma 2.2.1) now implies that $\{x_n\}$ converges to x. That is, we have a sequence $\{x_n\}$ of rational numbers that converges to our irrational number x.

Now consider $x \in \mathbb{R}$ irrational and let $\{x_n\}$ be a sequence of rational numbers converging to x. Note that since all the x_n are rational, $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$. Since f and g are continuous, we have

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

Hence $f(x) = g(x)$ for all $x \in \mathbb{R}$. □

Exercise 3.2.12. Let $f : \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.

Proof. We first note that there are no cluster points of $\mathbb{Z}$ in $\mathbb{R}$ (take any $0 < \epsilon < 1$ in the definition). Hence, by Proposition 3.2.2 (i), f is continuous at all points of $\mathbb{Z}$. □

Exercise 3.3.10. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Show that f has a fixed point, i.e. $\exists x \in [0, 1]$ such that $f(x) = x$.

Proof. Consider the function $g : [0, 1] \to [-1, 1]$, $g(x) = f(x) - x$. We would like to apply the intermediate value theorem to show that there is an $x \in [0, 1]$ such that $g(x) = 0$. Since f and the function $h(x) := x$ are both continuous, Proposition 3.2.5 (ii) tells us that g is also continuous.

Now, since $0 \le f(0)$ we have

$$g(0) = f(0) - 0 = f(0) \ge 0.$$

Similarly, since $f(1) \le 1$,

$$g(1) = f(1) - 1 \le 1 - 1 = 0.$$

So $g(1) \le 0 \le g(0)$ and by Theorem 3.3.8 (intermediate value theorem) there exists an $x \in [0, 1]$ such that

$$0 = g(x) = f(x) - x \implies f(x) = x.$$

□

5. Lebl Chapter 4 – The Derivative

Exercise 4.2.3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is a bounded function. Prove that f is a Lipschitz continuous function.

Proof. We need to show that there exists a number M such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Let $x, y \in \mathbb{R}$ and, without loss of generality, assume that $y < x$. Since f is differentiable on $[y, x]$, the mean value theorem (Theorem 4.2.4) implies that there is a $c \in [y, x]$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

Sjnce f' is bounded, there exists $M \in \mathbb{R}$ such that $|f'(c)| \leq M$ for all $c \in \mathbb{R}$. Thus

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

Since x and y were arbitrary, this shows that f is Lipschitz continuous. □

Exercise 4.2.5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that $f(x) = C$ for some constant C. *Hint: show that f is differentiable at all points and compute the derivative.*

Proof. If $f(x) = C$ for some constant C, then $f'(c) = 0$ for all $c \in \mathbb{R}$. We claim that $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is 0. We need to show that for all $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} \right| < \epsilon$$

whenever $|x - c| < \delta$.

Given $\epsilon > 0$, take $\delta = \epsilon$. Then

$$\left| \frac{f(x) - f(c)}{x - c} \right| = \frac{|f(x) - f(c)|}{|x - c|} \leq \frac{|x - c|^2}{|x - c|} = |x - c| < \delta = \epsilon.$$

Hence f is differentiable at all points and the derivative is 0. Proposition 4.2.6 now implies that f is constant. □

6. Lebl Chapter 5 – The Riemann Integral

Exercise 5.1.3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}$ of $[a, b]$ such that

$$\lim_{k\to\infty} \big(U(P_k, f) - L(P_k, f)\big) = 0.$$

Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k, f) = \lim_{k\to\infty} L(P_k, f).$$

Proof. Suppose that $\lim_{k\to\infty} \big(U(P_k, f)-L(P_k, f)\big) = 0$. By the definition of a limit, this means that for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $U(P_k, f) - L(P_k, f) < \epsilon$ for all $k > N$. Hence, for every $\epsilon > 0$, we can find a partition (P_k, for $k > N$ for example) of $[a, b]$ such that

$$U(P_k, f) - L(P_k, f) < \epsilon$$

and so f is integrable by Proposition 5.1.3.

Now for the second part, we know that

$$\overline{\int_a^b} f \le U(P_k, f), \quad \text{and } L(P_k, f) \le \underline{\int_a^b} f$$

for all k. Since f is integrable, $\overline{\int_a^b} f = \underline{\int_a^b} f$, so we have

$$0 \le U(P_k, f) - \overline{\int_a^b} f = U(P_k, f) - \underline{\int_a^b} f \le U(P_k, f) - L(P_k, f).$$

That is,

$$0 \le U(P_k, f) - \overline{\int_a^b} f \le U(P_k, f) - L(P_k, f)$$

and since $\lim_{k\to\infty} \big(U(P_k, f) - L(P_k, f)\big) = 0$, the squeeze theorem implies that

$$\lim_{k\to\infty} U(P_k, f) - \overline{\int_a^b} f = 0$$

$$\implies \lim_{k\to\infty} U(P_k, f) = \overline{\int_a^b} f = \int_a^b f.$$

Similarly, we have

$$0 \le \underline{\int_a^b} f - L(P_k, f) = \overline{\int_a^b} f \le U(P_k, f) - L(P_k, f)$$

and again, the squeeze theorem implies that $\lim_{k\to\infty} L(P_k, f) = \int_a^b f$. □